AD-A061 102 MICHIGAN UNIV ANN ARBOR DEPT OF INDUSTRIAL AND OPERA--ETC F/G 12/1
SOME PROBLEMS OF QUEUES WITH FEEDBACK.(U)
NOV 78 R L DISNEY, G R D'AVIGNON N00014-75-C-0492

UNCLASSIFIED NL

1 OF 1

AD
A061 102

END
DATE
FILMED
1-79
DDC

406 747





# Virginia Polytechnic Institute and State University

## Industrial Engineering and Operations Research

BLACKSBURG, VIRGINIA 24061

9 Technical rept.,



6 **Some Problems of Queues with Feedback**

by

10 Ralph L. Disney, Gilles R. d'Avignon
Gordon Professor
Virginia Polytechnic Institute & State University
Blacksburg, Virginia 24061

and

Gilles R. d'Avignon
Professor
Faculté des Sciences de l'Administration
Université Laval
Quebéc Gik 7P4
Canada

TECH REPORT
VTR - 78-2
11 Nov[redacted]1978

12 46p.

Paper prepared for presentation at the Colloquium on Point Processes and Queueing Theory, Keszthely, Hungary, September 4-8, 1978. Research for this paper has been supported by the Office of Naval Research under contracts N00014-67-A-0181-0046 (NR364-071), N00014-67-A-0181-0047 (NR042-296), and N00014-75-C-0492 (NR042-292).



15 NØØØ14-75-C-Ø492,
NØØØ14-67-A-Ø181-ØØ47

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER<br>Technical Report #VTR 78-2 | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br>SOME PROBLEMS OF QUEUES WITH FEEDBACK | | 5. TYPE OF REPORT & PERIOD COVERED<br>Technical Report |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>Ralph L. Disney and Gilles R. d'Avignon | | 8. CONTRACT OR GRANT NUMBER(s)<br>N00014-67-A-0181-0046<br>N00014-67-A-0181-0047<br>N00014-75-C-0492 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Queueing Network Research Project<br>Dept. of Industrial Engineering & Operations Res.<br>Virginia Polytechnic Inst. & State University<br>Blacksburg, Virginia 24061 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>NR364-071<br>NR042-296<br>NR042-292 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Director, Statistics and Probability Program<br>Mathematical & Information Sciences Division<br>800 N. Quincy St., Arlington, Va 22217 | | 12. REPORT DATE<br>November 1978 |
| | | 13. NUMBER OF PAGES<br>41 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>Unclassified |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release; Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Feedback

Queues

Markov Renewal Processes

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Queueing systems which include the possibility for a customer to return to the same server for additional service are called queueing systems with feedback. Such systems occur in computer networks, production networks, street traffic networks, neural networks, and the like. In spite of these potential applications, the study of such systems within the field of queueing theory is almost absent. ~~In~~ this paper ~~we will~~ present a few results for a broad class of queues with instantaneous feedback. Of particular interest here are queues with Markov renewal arrival processes, service times dependent on customer types and feedback mechanisms depending on queue length, increments, service times, customer types and previous histories of the feedback mechanism. It will be shown that networks as simple as Jackson networks with one

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Continuation of abstract.

server can present some formidable and unsolved problems. Special attention will be placed on these unsolved problems and new areas of queueing theory in need of research.

# SOME PROBLEMS OF QUEUES WITH FEEDBACK


## Abstract

Queueing systems which include the possibility for a customer to return to the same server for additional service are called queueing systems with feedback. Such systems occur in computer networks, production networks, street traffic networks, neural networks, and the like. In spite of these potential applications, the study of such systems within the field of queueing theory is almost absent.

In this paper we will present a few results for a broad class of queues with instantaneous feedback. Of particular interest here are queues with Markov renewal arrival processes, service times dependent on customer types and feedback mechanisms depending on queue length increments, service times, customer types and previous histories of the feedback mechanism. It will be shown that networks as simple as Jackson networks with one server can present some formidable and unsolved problems. Special attention will be placed on these unsolved problems and new areas of queueing theory in need of research.

# 1. INTRODUCTION

1.0. Background. In this paper we are concerned with queues with feedback. There are at least two interpretations that one can give to such a statement. We are concerned with a single server who serves an arrival stream. Upon completion of service, the customer may return for additional service or may depart the system, never to return. An alternate use of the words "queues with feedback" has been given in several publications by Kopocinski [1963], Koposinska [1967], and Kopocinska and Kopocinski [1972].

Problems of the type we describe seem to have first been studied, for M/G/1 queues, by Takacs [1963]. The problem became of interest to computer systems analysts under the name of round robin queueing and foreground-background queueing. Kleinrock [1975;1976] or Wyszewianski and Disney [1975] provide a thorough review of the computer modelling area including the feedback problem. Our interest in the problem originated with our overall interest in flow processes in queueing networks. Disney [1975] provides an overview of the queueing network problem and places the feedback problem in that context.

1.1. Outline. In this paper we will study a single server queue with feedback. The arrival process is a Markov renewal process on a finite state space--called an $MR_b$ process. The server process is a renewal process for each arrival "type" (b). The process is called a $G_b$ process. Upon completion of service, the unit feeds back or doesn't feed back, depending on previous feedback decisions, increments in the queue length process, service times just consumed, and customer type. Feedback always occurs instantly.

In section 2 we present the problem formally. In section 3 we will present results for the general problem stated above. Of particluar concern here is the queueing output process (see 2.1(B)). We then allow $MR_b$ to be a superposition of independent Poisson processes and explore the output process in detail. The busy period process is discussed in section 3. The queue length process is obtained marginally from the output process. We will simply discuss (without proof) that process, and provide some references on it in section 4. Research is still in progress on the departure process and waiting time process. The departure process is discussed in section 4. A brief discussion of the waiting time process is also considered there.

## 2. PRELIMINARIES

2.0. Introduction. We will be primarily concerned with queues with feedback in which the arrival process is a collection of mutually independent Poisson processes. In this section we will define the problem in somewhat more generality by allowing the arrival process to be a finite state Markov renewal process.

2.1. Formal Structure. Units arrive to a single server queue at epochs $\tau_1 < \tau_2 < \cdots < \tau_m < \cdots$ . If $Z_m$ denotes the type of the $m^{\underline{th}}$ arrival and $X_m = \tau_m - \tau_{m-1}$ represents the time between the $(m-1)^{\underline{st}}$ and the $m^{\underline{th}}$ arrivals. Then the sequence $Z_m, X_m$ will be called the arrival process. The different types of arrivals form the state space of the arrival process.

<u>Assumption A</u>. The arrival process $\{Z_m, X_m\}$ is an irreducible, aperiodic Markov renewal process over the state space $\{1,2,\cdots,b\}$. Moreover, the stationary transition probability $A_{hk}(x) = \Pr\{Z_m = k, X_m \leq x | Z_{m-1} = h\}$, is such for all $x \geq 0$, $A_{hk}(0+) = 0$, and $\sum_{k=1}^{b} A_{hk}(\infty) = 1$ for $h = 1,2,\cdots b$, $k = 1,2,\cdots,b$.

Throughout this paper sysbols such as $\sum_k^b$ or $\prod_c^b$ is used to denote $\sum_{k=1}^b$ or $\prod_{c=1}^b$.

Let the probability distribution of the interarrival time $X_m$ be

$$B_h(x) = \sum_k^b A_{hk}(x) \text{ for } h = 1,2,\cdots,b.$$

Furthermore, let $a_{hk} = A_{hk}(\infty)$ be the transition probabilities of the underlying Markov chain $\{Z_m\}$.

We will call this arrival process an $MR_b$ process to denote that it is a Markov renewal process with b ($<\infty$) states. In most of our work, we will be concerned with the special case in which the arrival process is the superposition of b mutually independent Poisson processes with parameters $\lambda_k (k = 1,2,\cdots,b)$. In this cases, we will call the arrival process an $M_b$ process.

Let $S_n$ be the length of the $n^{\underline{th}}$ service and $Z'_n$ be the type of the unit which received the $n^{\underline{th}}$ service. The sequence $\{S_n\ Z'_n\}$ will be called the <u>service process</u>. Notice that $Z_n \neq Z'_n$ in general. $Z_n$ is the type of the $n^{\underline{th}}$ arrival. $Z'_n$ is the type of customer receiving the $n^{\underline{th}}$ service.

<u>Assumption B</u>. The service process $\{S_n | Z'_n\}$ is a sequence of mutually conditionally independent random variables over the state space $\{1,2,\cdots,b\}$. That is,

$$\Pr[S_{n_1} \leq s_{n_1}, S_{n_2} \leq s_{n_2}, \cdots, S_{n_k} \leq s_{n_k} | Z'_{n_1}, Z'_{n_2}, \cdots, Z'_{n_k}]$$

$$= \Pr[S_{n_1} \leq s_{n_1} | Z'_{n_1}] \Pr[S_{n_2} \leq s_{n_2} | Z'_{n_2}] \cdots \Pr[S_{n_k} \leq s_{n_k} | Z'_{n_k}].$$

for all $n_i$ in $0,1,2,\cdots$ and all $s_i \in \mathbb{R}_+$ and $n_i \neq n_j$. Moreover, $H_r(x) = \Pr\{S_n \leq x | Z'_n = r\}$, is such that $H_r(0+) = 0$, $H_r(\infty) = 1$, $\bar{S}_r = E[S_n | Z'_n = r] < \infty$ and $\bar{V}_r = V[S_n | Z'_n = r] < \infty$ for $r = 1,2,\cdots,b$.

We denote the service process by $G_b$. When $b = 1$, the service process becomes renewal (G) and the probability distribution is then noted by $H(\cdot)$.

Let service completions occur at epochs $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots$. $\{t_n\}$ is the sequence of output epochs. Let $\underline{N}_n = \underline{N}(t_n+)$ be the vector of queue lengths at $t_n+$. The entry $N_n^c$ denotes the number of type-c units in the system, excluding the fed back unit it feeds back.

Associated with the $n^{\underline{th}}$ output, the random variable $Y_n$ is defined as follows:

$$Y_n = \begin{cases} 0\,, & \text{if the } n^{\underline{th}} \text{ output departs,} \\[2em] v\,, & \text{if the } n^{\underline{th}} \text{ output feeds back into a type-v customer} \\ & (v = 1,2,\cdots,b). \end{cases}$$

The value that $Y_n$ takes is referred to as the <u>output status</u> of the $n^{\underline{th}}$ output. The state of the system at $t_n+$ is $Y_n = u$ and $\underline{N}_n = \underline{i} = [i_c]$, with the understanding that the queue length of type-c units is $\delta_{uc} + i_c$. The sequence $\{Y_n\}$ is called the <u>switching process</u> and is determined by the following

<u>Assumption C</u>. The output status depends only on the previous output status, the increments in the queue lengths, the amount of service the unit has just received and the type of unit receiving the service. That is,

$$\Pr\{Y_n = v \mid Y_{n-1} = u,\ N_n = j,\ N_{n-1} = i,\ S_n = y,\ Z'_n = r\} = \begin{cases} p_{0v}(\underline{j}, r; y), & \text{if } u = 0 \ \& \ \underline{i} = \underline{0}, \\ p_{uv}(\underline{e}, r; y), & \text{otherwise}; \end{cases}$$

where $\underline{e} = [j_c + \delta_{rc} - \delta_{uc} - i_c]$.

We assume that every customer eventually leaves the system. Thus we have an open network in the usual sense. If every customer departs the system after one service time then the following results are relevant to $MR_b/G_b/1$ queues without feedback. In that case the departure process and the output process are identical. Thus results pertaining to outputs are in reality results for the departure process.

Concerning the time between service completions and the re-entry of a fed back unit, we have

Assumption D. There is no delay in feeding back a unit.

After a service completion, the next unit to be served is chosen according to the following service discipline.

Assumption E. Whenever there is no unit in the system just after the $(n-1)^{\underline{st}}$ output, the server remains idle until the first arrival. This arrival determines the type of unit receiving the $n^{\underline{th}}$ service, $Z'_n$. In the case where the system does not become empty after $t_{n-1}$, the next unit to be served is chosen from the non-empty queue having the lowest type index. That is, $Z'_n = r$ where $r = \min\{c : \delta_{uc} + i_c > 0,\ c = 1, 2, \cdots, b\}$. Among units of the same type, the service discipline is first-come, first served.

The additional three assumptions are also made.

Assumption F. At epoch $t_0$, a service completion occurred and the corresponding unit left the system. Moreover, the initial queue length probabilities are given to be $\Pr\{\underline{N}_0 = \underline{i} | Y_0 = 0\} = q_0(\underline{i})$.

Assumption G. There exists an infinite capacity for each type of unit in front of the server.

Assumption H. There is only one server.

The queueing system with feedback to be referred to in sections 2-4 is the model which satisfies the above assumptions. Therefore, we give

Definition 2.1. An $MR_b/G_b/1$ queue with feedback is a queueing system satisfying assumptions A through H.

## 3. QUEUEING PROPERTIES

3.0. Introduction. In section 2 we established the general framework of the model. In this section we will explore some of its properties.

Though the assumptions of the model in section 2 are more general than we will pursue in detail, one can obtain some results from them. These results will occupy our attention in section 3.1. In the remainder of the section, we will turn to the special case of superposed Poisson processes and study the output process and busy period process in detail. The queue length process will be only mentioned. The departure process can be obtained from results herein by filtering the output process on $Y_n = 0$. Some details for cases in which $\{Y_n\}$ does not depend on $Z'_n$ are given in d'Avignon and Disney [1978].

The sojourn times problem can be given a Markov renewal structure, but it seems to be quite difficult to work with. The only complete results we know of for this process are those given in Takacs [1963] for the M/G/1 queue with $\{Y_n\}$ a Bernoulli process.

3.1. The Output Process. The queueing system is considered at output epochs $t_n$. Denote by $K_n$ the type of the last arrival before $t_n$ and by $U_n$ the time since the last arrival measured from $t_n$. Moreover, let $O_n = t_n - t_{n-1}$, $O_o = 0$ be the output interval between output epochs $t_{n-1}$ and $t_n$. The sequence $\{Y_n, \underline{N}_n, K_n, U_n, O_n\}$ is called the output process. The state space of such random process is the cross product of a denumerable set and the non-negative real numbers.

Denote by $I_n$ the idle period preceeding the $n^{\underline{th}}$ output. Notice that whenever the system does not become empty after the service completion $t_{n-1}$, $I_n = 0$.

Later on, we will need to establish relationships among the random variables $\underline{N}_n, K_n, U_n, S_n$ and $I_n$ for different output epochs $t_n$'s. Figures 3.1 and 3.2 below illustrate some of the relationships.

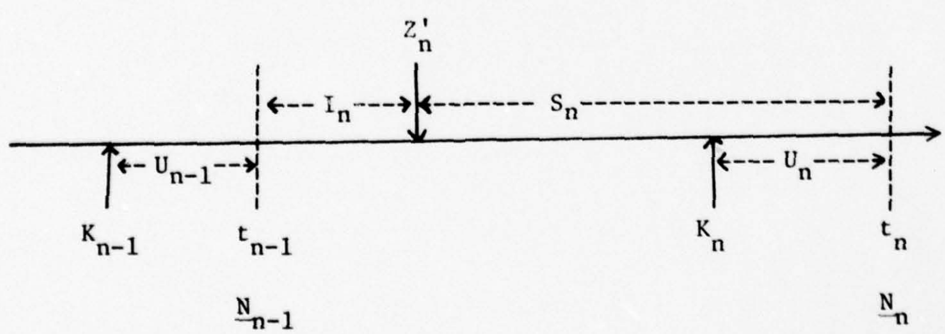


Figure 3.1. The process with an idle period.

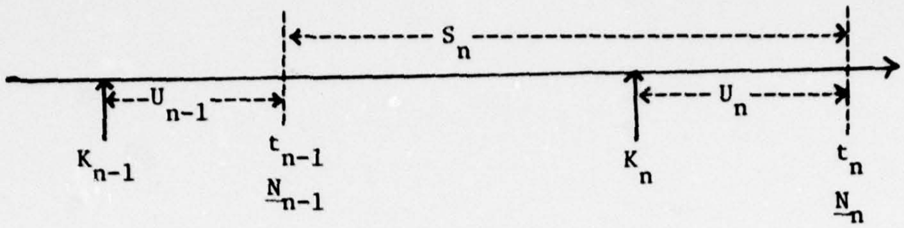


Figure 3.2. The process without an idle period.

<u>Proposition 3.1</u>. The random variables $\underline{N}_n$, $K_n$ and $U_n$ depend only on $S_n$, $Z'_n$, $Y_{n-1}$, $\underline{N}_{n-1}$, $K_{n-1}$, and $U_{n-1}$.

<u>Proof</u>. If $Y_{n-1} = u$, the total number of units in the system just after $t_{n-1}$, is given by

$$B = \sum_{c}^{b} (\delta_{uc} + N^{c}_{n-1}).$$

Therefore, the system is idle or not depending on this last quantity; consequently, one considers two cases.

i) $B = 0$: assumption A and Figure 3.1 make apparent that $\underline{N}_n$, $K_n$ and $U_n$ depend only on $S_n$ and $Z'_n$.

ii) $B > 0$: in this case the service starts immediately on a new unit. According to assumption E, this new unit is of type $Z'_n = r$ where

$$r = \min\{c\colon \delta_{uc} + N^{c}_{n-1} > 0,\ \ c = 1,2,\cdots,b\}.$$

Again, using assumption A and from Figure 3.2, it is clear that $\underline{N}_n$, $K_n$ and $U_n$ depend only on $S_n$, $Y_{n-1}$, $\underline{N}_{n-1}$, $K_{n-1}$, and $U_{n-1}$. $\square$

Denote by $f_{hk}(\underline{e},y,t|s)$ the joint probability of $\underline{N}_n$, $K_n$ and $U_n$. In words, this is the probability for a set $\underline{e} = [e_c]$ of arrivals in an interval of length y-t with the last one of type k and no other arrival during a length t given that prior to that interval of length y-t the last arrival was of type h and occurred at a distance s from the beginning of the interval of length y-t. Let

$$i = e_1 + e_2 + \cdots e_b,$$

be the total number of arrivals in the length y-t. Then,

$$f_{hk}(\underline{e},y,t|s) = \Pr\{Z_{m+1} = k,\ X_{m+1} + \cdots + X_{m+i} = y-t+s$$

$$\text{with } e_c \text{ type-c arrivals, } X_{m+i+1} > t | Z_m = h,\ X_{m+1} > s\}.$$

For the case where $\underline{e} = \underline{0}$, the above probability is found to be

$$f_{hk}(\underline{0},y,t|s) = \delta_{hk}\delta(s + y-t)\,\frac{1 - B_h(t)}{1 - B_h(s)}\ ,$$

since there is no arrival in the length y-t, one must have $t = s + y$ and the last arrival of type h.

For the case where $\underline{e} \neq \underline{0}$, there is some arrival and t must be smaller than or equal to y. Otherwise, $f_{hk}(\underline{e},y,t|s) = 0$.

Proposition 3.2. Whenever $I_n \neq 0$, it depends only on $Z_n'$, $K_{n-1}$ and $U_{n-1}$; and its probability distribution is given by

$$\Pr\{I_n \leq x | Z_n' = r,\ K_{n-1} = h,\ U_{n-1} = s\} = \frac{A_{hr}(s+x) - A_{hr}(s)}{a_{hr} - A_{hr}(s)}\ ,$$

for $r,h = 1,2,\cdots,b$ and $x,s \geq 0$.

Proof. From assumption A, one has that $I_n$ depends only on $Z'_n$, $K_{n-1}$ and $U_{n-1}$. Furthermore, it follows that

$$\text{(1)} \qquad Pr\{I_n \leq x \mid Z'_n = r,\ K_{n-1} = h,\ U_{n-1} = s\} = \frac{Pr\{Z'_n = r,\ U_{n-1} = s,\ I_n \leq x \mid K_{n-1} = h\}}{\{Pr\ Z'_n = r,\ U_{n-1} = s \mid K_{n-1} = h\}} .$$

By considering the arrival process defined in assumption A, one finds

$$Pr\{Z'_n = r,\ U_{n-1} = s\ I_n \leq x \mid K_{n-1} = h\} = A_{hr}(s+x) - A_{hr}(s) ,$$

and

$$Pr\{Z'_n = r,\ U_{n-1} = s \mid K_{n-1} = h\} = a_{hr} - A_{hr}(s) ;$$

which after substitution in (1), gives the desired result. □

Proposition 3.3. The type of the $n^{\underline{th}}$ output, $Z'_n$, depends only on $Y_{n-1}$, $\underline{N}_{n-1}$, $K_{n-1}$ and $U_{n-1}$; and its probability distribution is given by

$$Pr\{Z'_n = k \mid Y_{n-1} = u,\ \underline{N}_{n-1} = \underline{i},\ K_{n-1} = h,\ U_{n-1} = s\} = \begin{cases} \dfrac{a_{hk} - A_{hk}(s)}{1 - B_h(s)}, & \text{if } u = 0\ \&\ \underline{i} = \underline{0}, \\ \delta_{rk} & \text{otherwise;} \end{cases}$$

where $r = \min\{c\colon \delta_{uc} + i_c > 0,\ \ c = 1,2,\cdots,b\}$.

Proof. Again, there are two cases to consider:

i) $u = 0$ and $\underline{i} = \underline{0}$: in this case the system becomes idle just after the output epoch $t_{n-1}$. From assumption A, the next arrival which becomes the $n^{\underline{th}}$ output depends only on $K_{n-1}$ and $U_{n-1}$. Furthermore, its probability distribution is

(1) $$\Pr\{Z'_n = k \mid K_{n-1} = h,\ U_{n-1} = s\} = \frac{\Pr\{Z'_n = k,\ U_{n-1} = s \mid K_{n-1} = h\}}{\Pr\{U_{n-1} = s \mid K_{n-1} = h\}} \ .$$

By the arrival process of assumption A, one has

(2) $$\Pr\{Z'_n = k,\ U_{n-1} = s \mid K_{n-1} = h\} = a_{hk} - A_{hk}(s) \ ,$$

and

(3) $$\Pr\{U_{n-1} = s \mid K_{n-1} = h\} = 1 - B_h(s).$$

Substituting equations (2) and (3) into equation (1), one obtains the first part of the desired result.

ii) $u \neq 0$ or $\underline{i} \neq \underline{0}$: in this case the system remains busy after $t_{n-1}$ and the next unit to be served is chosen with respect to assumption E. Therefore, the type of the $n^{\underline{th}}$ output is $Z'_n = r$, where

$$r = \min\{c:\ \delta_{uc} + i_c > 0,\quad c = 1,2,\cdots,b\}.$$

Then, the second part of the theorem follows. □

Now, one has the following important result.

<u>Theorem 3.4</u>. The output process $\{Y_n, \underline{N}_n, K_n, U_n, O_n\}$ is Markov renewal over the state space

$$\{0,1,\cdots,b\}\ X\{0,1,2,\cdots\}^b\ X\{1,2,\cdots,b\}\ X[0,\infty).$$

<u>Proof</u>. By definition, the range of the random variable $Y_n$ is $\{0,1,\cdots,b\}$ and by considering the range of the other random variables, one obtains the state space specified.

The output interval $0_n$ can be written as

$$0_n = \begin{cases} I_n + S_n, & \text{if } \sum_c^b (\delta_{uc} + N_{n-1}^c) = 0, \\ S_n, & \text{otherwise;} \end{cases}$$

where u is the value taken by the random variable $Y_{n-1}$.

The above relationship gives two cases:

i) $\sum_c^b (\delta_{uc} + N_{n-1}^c) = 0$: this case corresponds to the situation where the $(n-1)^{\underline{st}}$ output departs from the system leaving the server idle. Therefore, a unit of unknown type must arrive to initiate the service period $S_n$. Let $Z_m$ be the type of this new arriving unit. This unit will also be the $n^{\underline{th}}$ output; therefore $Z_n' = Z_m$. and one has the following:

$$(1) \quad \Pr\{Y_n, \underline{N}_n, K_n, U_n, 0_n \le x \mid Y_{n-1}, \cdots,; \underline{N}_{n-1}, \cdots; K_{n-1}, \cdots; U_{n-1}, \cdots; 0_{n-1}, \cdots\}$$

$$= \int_0^x \sum_k^b \Pr\{Y_n \mid Y_{n-1}, \cdots; \underline{N}_n, \cdots; K_n, \cdots; U_n, \cdots; 0_{n-1}, \cdots; \quad I_n + S_n \le x,\ S_n = y,\ Z_n' = k\}$$

$$\cdot \Pr\{\underline{N}_n, K_n, U_n \mid Y_{n-1}, \cdots; \underline{N}_{n-1}, \cdots; K_{n-1}, \cdots; U_{n-1}, \cdots; 0_{n-1}, \cdots; \quad I_n + S_n \le x,\ S_n = y,\ Z_n' = k\}$$

$$\cdot \Pr\{I_n + S_n \le x \mid Y_{n-1},\cdots; \underline{N}_{n-1},\cdots; K_{n-1},\cdots; U_{n-1},\cdots; 0_{n-1},\cdots;$$

$$S_n = y,\ Z'_n = k\}$$

$$\cdot \Pr\{S_n = y \mid Y_{n-1},\cdots; \underline{N}_{n-1},\cdots; K_{n-1},\cdots; U_{n-1},\cdots; 0_{n-1},\cdots; Z'_n = k\}$$

$$\cdot \Pr\{Z'_n = k \mid Y_{n-1},\cdots; \underline{N}_{n-1},\cdots; K_{n-1},\cdots; U_{n-1},\cdots; 0_{n-1},\cdots\}dy.$$

Using assumption C, the first probability term on the right hand side of equation (1) is $\Pr\{Y_n \mid Y_{n-1},\underline{N}_n,\underline{N}_{n-1},\ S_n = y,\ Z'_n = k\}$. Together with propositions 3.1, 3.2, 3.3, and assumption B, equation (1) becomes

(2) $$\Pr\{Y_n,\underline{N}_n,K_n,U_n,0_n \le x \mid Y_{n-1},\cdots; \underline{N}_{n-1},\cdots; K_{n-1},\cdots; U_{n-1},\cdots; 0_{n-1},\cdots\}$$

$$= \int_0^x \sum_k^b \Pr\{Y_n \mid Y_{n-1},\underline{N}_n,\underline{N}_{n-1},\ S_n = y,\ Z'_n = k\}$$

$$\cdot \Pr\{\underline{N}_n,K_n,U_n \mid S_n = y,\ Z'_n = k\}$$

$$\cdot \Pr\{I_n \le x-y \mid K_{n-1},U_{n-1},\ S_n = y,\ Z'_n = k\}$$

$$\cdot \Pr\{S_n = y \mid Z'_n = k\}$$

$$\cdot \Pr\{Z'_n = k \mid K_{n-1},U_{n-1}\}\ dy\ .$$

Equation (2) can now be rewritten as

(3) $$\Pr\{Y_n,\underline{N}_n,K_n,U_n,0_n \le x \mid Y_{n-1},\cdots; \underline{N}_{n-1},\cdots; K_{n-1},\cdots; U_{n-1},\cdots; 0_{n-1},\cdots\}$$

$$= \Pr\{Y_n,\underline{N}_n,K_n,U_n,0_n \le x \mid Y_{n-1},\underline{N}_{n-1},K_{n-1},U_{n-1}\}.$$

ii) $\sum_{c}^{b}(\delta_{uc} + N_{n-1}^{c}) > 0$: in this case, there is no idle period and the service immediately starts on a new unit of type $Z_n' = r$ where

$$r = \min\{c: \delta_{uc} + N_{n-1}^{c} > 0,\ c = 1,2,\cdots,b\}.$$

Thus,

$$(4) \quad \Pr\{Y_n, \underline{N}_n, K_n, U_n, O_n \leq x | Y_{n-1}, \cdots; \underline{N}_{n-1}, \cdots; K_{n-1}, \cdots; U_{n-1}, \cdots; O_{n-1}, \cdots\}$$

$$= \int_0^x \Pr\{Y_n | Y_{n-1}, \cdots; \underline{N}_n, \cdots; K_n, \cdots; U_n, \cdots; O_{n-1}, \cdots;\ S_n = y,\ Z_n' = r\}$$

$$\cdot \Pr\{\underline{N}_n, K_n, U_n | Y_{n-1}, \cdots; \underline{N}_{n-1}, \cdots; K_{n-1}, \cdots; O_{n-1}, \cdots; S_n = y,\ Z_n' = r\}$$

$$\cdot \Pr\{S_n = y | Y_{n-1}, \cdots; \underline{N}_{n-1}, \cdots; K_{n-1}, \cdots; U_{n-1}, \cdots; O_{n-1}, \cdots;\ Z_n' = r\}\, dy,$$

using the fact that $O_n = S_n$ in this case. In the above equation, the first probability term on the right hand side is simply

$$\Pr\{Y_n | Y_{n-1}, \underline{N}_{n-1}, \underline{N}_n,\ S_n = y,\ Z_n' = r\}\ ,$$

using assumption C. The second term is simplified upon using proposition 3.1 and for the third term, one notes that $S_n$ depends only on $Z_n' = r$ from assumption B.

Hence, equation (4) becomes

$$(5) \quad \Pr\{Y_n, \underline{N}_n, K_n, U_n, O_n \leq x | Y_{n-1}, \cdots; \underline{N}_{n-1}, \cdots; K_{n-1}, \cdots; U_{n-1}, \cdots; O_{n-1}, \cdots\}$$

$$= \int_0^x \Pr\{Y_n | Y_{n-1}, \underline{N}_{n-1}, \underline{N}_n,\ S_n = y,\ Z_n' = r\}$$

$$\cdot \Pr\{\underline{N}_n, K_n, U_n | Y_{n-1}, \underline{N}_{n-1}, K_{n-1}, U_{n-1},\ S_n = y,\ Z_n' = r\}$$

$$\cdot \Pr\{S_n = y | Z_n' = r\}\, dy\ .$$

Equation (5) can be rewritten as equation (3) above and one concludes that $\{Y_n, \underline{N}_n, K_n, Y_n, 0_n\}$ has the Markov renewal property. $\square$

<u>Corollary 3.5</u>. The stationary transition probabilities for the Markov renewal process $\{Y_n, \underline{N}_n, K_n, U_n, 0_n\}$ are given by

$$A_{uv}[(\underline{i},h,s),(\underline{j},k,t);\,x]$$

$$= \Pr\{Y_n = v, \underline{N}_n = \underline{j},\ K_n = k,\ U_n = t, 0_n \leq x \mid Y_{n-1} = u, \underline{N}_{n-1} = \underline{i}, K_{n-1} = h,$$

$$U_{n-1} = s\},$$

where

$$A_{uv}[(\underline{i},h,s),(\underline{j},k,t);x] = \begin{cases} \displaystyle\int_0^x \sum_a^b p_{0v}(\underline{j},a;y)\, f_{ak}(\underline{j},y,t \mid 0) \cdot \left( \frac{A_{ha}(s+x-y) - A_{ha}(s)}{1 - B_h(s)} \right) dH_a(y), & \text{if } u = 0 \ \&\ \underline{i} = \underline{0}, \\ 0 \quad , & \text{if } e_c < 0 \text{ for some } c, \\ \displaystyle\int_0^x p_{uv}(\underline{e},r;y)\, f_{hk}(\underline{e},y,t \mid s)\, dH_r(y), & \text{otherwise;} \end{cases}$$

with $\underline{e} = [e_c]$ and $e_c = j_c + \delta_{rc} - \delta_{uc} - i_c$; for $r = \min\{c\colon \delta_{uc} + i_c > 0,$ $c = 1,2,\cdots,b\}$.

<u>Proof</u>. From theorem 3.4, there are two cases to consider;

i) $u = 0$ and $\underline{i} = \underline{0}$: here equation (2) of theorem 3.4 can be applied and after using assumption C, one finds

$$\Pr\{Y_n = v \mid Y_{n-1} = 0, \underline{N}_{n-1} = 0, \underline{N}_n = \underline{j}, S_n = y, Z'_n = a\} = p_{0v}(\underline{j}, a; y).$$

And

$$\Pr\{\underline{N}_n = \underline{j}, K_n = k, U_n = t \mid S_n = y, Z'_n = a\} = f_{ak}(\underline{j}, y, t \mid 0).$$

Furthermore, from proposition 3.2,

$$\Pr\{I_n \leq x-y \mid Z'_n = a,\ K_{n-1} = h,\ U_{n-1} = s\} = \frac{A_{ha}(s+x-y) - A_{ha}(s)}{a_{ha} - A_{ha}(s)} \quad ,$$

and from assumption B

$$\Pr\{S_n = y \mid Z'_n = a\}\, dy = dH_a(y).$$

Finally, from proposition 3.3,

$$\Pr[Z'_n = a \mid K_{n-1} = h,\ U_{n-1} = s] = \frac{a_{ha} - A_{ha}(s)}{1 - B_h(s)} \quad .$$

Then

$$A_{0v}[(\underline{0}, k, s), (\underline{j}, h, t); x] = \int_0^x \sum_a^b p_{0v}(\underline{j}, a; y)\, f_{ah}(\underline{j}, y, t \mid 0)$$

$$\cdot \left( \frac{A_{ha}(s+x-y) - A_{ha}(s)}{1 - B_h(s)} \right) dH_a(y) \ .$$

ii) $u > 0$ or $\underline{i} \neq \underline{0}$: here one computes

$$r = \min\{c:\ \delta_{uc} + i_c > 0,\quad c = 1, 2, \cdots, b\},$$

for the type of unit to be served and one can apply equation (5) of theorem 3.4 to get

$$(1) \qquad A_{uv}[\underline{i},h,s),(\underline{j},k,t);x] = \int_0^x p_{uv}(\underline{e},r;y)\, f_{hk}(\underline{e},y,t|s)\, dH_r(y),$$

where $\underline{e} = [e_c]$ with $e_c = j_c + \delta_{uc} - \delta_{uc} - i_c$. If some components $j_c + \delta_{rc} - \delta_{uc} - i_c$ are negative, the transition probabilities given by equation (1) are zero. □

Proposition 3.6. The joint probability distribution of $Z'_n$ and $0_n$ depends only on $Y_{n-1}$, $\underline{N}_{n-1}$, $K_{n-1}$, $U_{n-1}$, and is given by

$$\Pr\{Z'_n = k,\ 0_n \le x | Y_{n-1} = u,\ \underline{N}_{n-1} = \underline{i},\ K_{n-1} = h,\ U_{n-1} = s\}$$

$$= \begin{cases} \displaystyle\int_0^x \left\{ \frac{A_{hk}(s+x-y) - A_{hk}(s)}{1 - B_h(s)} \right\} dH_k(y) & \text{if } u = 0 \text{ and } \underline{i} = \underline{0}, \\ \delta_{rk} H_k(s) & \text{if } u > 0 \text{ or } \underline{i} \ne \underline{0}; \end{cases}$$

where $r = \min\{c\colon \delta_{uc} + i_c > 0,\ c = 1,2,\cdots,b\}$.

Proof. Using theorem 3.4 and proposition 3.3, one can establish the correct dependence. The explicit probability is found by using corollary 3.5; note that in the case $u = 0$ and $\underline{i} = \underline{0}$, a unit of type-k must arrive. □

3.2. The Case $\underline{MR}_b = \underline{M}_b$. The previous general results can be expanded on or particularized in several directions. For example, by allowing $b = 1$, one obtains results for GI/G/1 queues with feedback. If further there is no feedback, one obtains a characterization of the GI/G/1 departure process previously

obtained by Vlach and Disney [1969]. Rather than pursue those directions, we now particularize by allowing $MR_b = M_b$. We start with a rather obvious lemma.

Lemma 3.7. $MR_b = M_b$ if and only if the arrival process stated in assumption A has the stationary transition probability distributions given by

$$A_{hk}(x) = \alpha_k(1 - e^{-x\sum\lambda}),$$

where $\alpha_k = \lambda_k/\sum\lambda$ for $h,k = 1,2,\cdots,b$ and $x \geq 0$. $\sum\lambda$ represents the summation of $\lambda_k$ for k going from 1 to b.

Proof. The necessary part is easily obtained by considering reaults from McFadden [1962]. The sufficiency is first proved for b = 2; let $T_k$ be the time interval between two arrivals of type-k units for $k = 1,2$. If N denotes the number of arrivals necessary to obtain a second type-k unit, then N is a random variable with the range $\{1,2,\cdots\}$.

Now, the probability distribution of $T_1$ is derived as follows:

$$(1) \qquad F_{T_1}(x) = \sum_{n=1}^{\infty} \Pr\{T_1 \leq x, N = n\},$$

$$= \sum_{n=1}^{\infty} F_{T_1}(x;n).$$

Assumption A is used to find

$$(2) \qquad F^*_{T_1}(s;n) = \begin{cases} A^*_{11}(s), & \text{if } n = 1, \\ A^*_{12}(s)[A^*_{22}(s)]^{n-2} A^*_{21}(s), & \text{if } n = 2,3,\cdots; \end{cases}$$

where

(3) $$A^*_{hk}(s) = \frac{\lambda_k}{\lambda_1 + \lambda_2 + s},$$

for $h,k = 1,2$.

Thus, using equations (2) and (3) into equation (1), one obtains

$$F^*_{T_1}(s) = \frac{\lambda_1}{\lambda_1 + s}$$

that is, $T_1$ is exponentially distributed with parameter $\lambda_1$. Similarly, one can show that $T_2$ is also exponentially distributed with parameter $\lambda$.

Since the stationary transition probability distribution, $A_{hk}(\cdot)$ does not depend on h, the types of two consecutive arrivals are independent. Also, since $T_k(k=1,2)$ has the usual forgetfulness property, the arrival process is $M_2$.

For the case $b > 2$, one can form two groups: $\lambda_1$ and $\lambda_2' = \lambda_2 + \lambda_2 + \cdots + \lambda_b$ and use the method above to show that $T_k(k=1,2,\cdots b)$ has the usual forgetfulness property. Therefore, the arrival process is $M_b$. □

Proposition 3.8. If $MR_b = M_b$, then

$$f_{hk}(\underline{e},y,t|s) = \begin{cases} \delta_{hk}\delta(y+s-t)\exp(-(t-s)\sum\lambda), & \text{if } \underline{e}=\underline{0}, \\ \lambda_k \exp(-t\sum\lambda)\prod_r \dfrac{\exp(-\lambda_r(y-t))[\lambda_r(y-t)]^{e_r-\delta_{rk}}}{(e_r-\delta_{rk})!}, & \\ \qquad \text{if } \underline{e}=\underline{0}; \end{cases}$$

for $\underline{e} = [e_c]$ and $e_c = 0,1,2,\cdots$ .

<u>Proof</u>. There are two cases to consider: for $\underline{e} = \underline{0}$, one uses lemma 3.7 and obtains

$$f_{hk}(\underline{e},y,t|s) = \delta_{hk}\delta(t-s-y)\exp(-(t-s)\textstyle\sum\lambda) .$$

For $\underline{e} \neq \underline{0}$, one needs a set $\underline{e} = [e_c]$ of arrivals, where $e_c$ denotes the number of type-c arrivals. The number of arrivals of each type in an interval of length $y-t$ is Poisson distributed, and since one wants a type-k arriving at the end of the interval of length $y-t$ and no arrival for a length t, we have

$$f_{hk}(\underline{e},y,t|s) = \left(\prod_{\substack{r=1\\ r\neq k}}^{b} \frac{\exp(-\lambda_r y)[\lambda_r(y-t)]^{e_r}}{e_r!}\right) \frac{\lambda_k^{e_k}(y-t)^{e_k-1}}{(e_k-1)!}\exp(\lambda_k(y-t)) ,$$

which is the desired result. □

<u>Proposition 3.9.</u> If $MR_b = M_b$, then the output process, $\{Y_n, \underline{N}_n, 0_n\}$ is a Markov renewal process and its stationary transition probabilities are given by

$$A_{uv}(\underline{i},\underline{j};x) = \Pr\{Y_n = v, \underline{N}_n = \underline{j}, 0_n \leq x | Y_{n-1} = u, \underline{N}_{n-1} = \underline{i}\},$$

where

$$A_{uv}(\underline{i},\underline{j};x) = \begin{cases} \displaystyle\int_0^x \sum_h^b \alpha_h p_{0v}(\underline{j},h;y)(1-\exp(-(x-y)\textstyle\sum\lambda))\displaystyle\prod_c^b \frac{\exp(-\lambda_c y)(\lambda_c y)^{j_c}}{j_c!} \\ \qquad\qquad \cdot\, dH_h(y) , \qquad \text{if } u=0 \text{ and } \underline{i}=\underline{0}, \\ 0 , \qquad\qquad\qquad\qquad \text{if } e_c < 0 \text{ for some c}, \\ \displaystyle\int_0^x p_{uv}(\underline{e},r;y)\prod_c^b \frac{\exp(-\lambda_c y)(\lambda_c y)^{e_c}}{e_c!}\, dH_r(y), \text{ otherwise,} \end{cases}$$

with $\underline{e} = [e_c]$, $e_c = j_c + \delta_{rc} - \delta_{uc} - i_c$; for $r = \min\{c\colon \delta_{uc} + i_c > 0,\ \ c = 1,2,\cdots,b\}$.

Proof. Since one has the forgetfulness property whenever the arrival process is composed of independent Poisson streams, the information on the arrival process carried by $K_n$ and $U_n$ becomes unnecessary and can be removed. Hence, in this case, the output process $\{Y_n, \underline{N}_n, K_n, U_n, 0_n\}$ can be reduced to $\{Y_n, \underline{N}_n, 0_n\}$. This reduction is accomplished by computing the transition probabilities

$$(1) \qquad A_{uv}[(\underline{i},h,s)(\underline{j},\cdot,\cdot);x] = \sum_k^b \int_0^\infty A_{uv}[(\underline{i},h,s),(\underline{j},k,t);x]dt.$$

Consider two cases:

i) $u = 0$ and $\underline{i} = \underline{0}$: from corollary 3.5, equation (1) becomes

$$A_{0v}[(\underline{0},h,s),(\underline{j},\cdot,\cdot);x] = \int_0^x \sum_k^b \sum_a^b \int_0^\infty p_{0v}(\underline{j},a;y)\, f_{ak}(\underline{j},y,t|0)\, dt \cdot \left( \frac{A_{ha}(s+x-y) - A_{ha}(s)}{1 - B_h(s)} \right) dH_a(y) .$$

Thus, one obtains

$$\frac{A_{ha}(s+x-y) - A_{ha}(s)}{1 - B_h(s)} = \alpha_a[1 - \exp(-(x-y)\textstyle\sum\lambda)] .$$

Also, from proposition 3.10, one finds

$$(2) \qquad \sum_k^b f_{ak}(\underline{j},y,t|0) = \begin{cases} \delta(y-t)\exp(-t\Sigma\lambda), & \underline{j} = \underline{0}, \\ R\exp(-y\Sigma\lambda) \left( \prod_c^b \dfrac{\lambda_c^{j_c}}{j_c!} \right) (y-t)^{R-1}, & \underline{j} \neq \underline{0}; \end{cases}$$

where $R = \sum_{c}^{b} j_c$.

Then

$$A_{0v}[\underline{0},h,s),(\underline{0},\cdot,\cdot);x]$$

$$= \int_0^x \sum_a^b \alpha_a p_{0v}(\underline{0},a,y)(1 - \exp(-(x-y)\sum\lambda))\exp(-(y)\sum\lambda)\, dH_a(y),$$

and for $\underline{j} \neq \underline{0}$,

$$A_{0v}[\underline{0},h,s),(\underline{j},\cdot,\cdot);x]$$

$$= \int_0^x \sum_a^b \alpha_a p_{0v}(\underline{j},a,y)(1 - \exp(-(x-y)\sum\lambda))\prod_c^b \frac{\exp(-(\lambda_c y)(\lambda_c y)^{j_c}}{j_c!}\, dH_a(y).$$

Since the right hand side does not depend on h and s, one can write

(3) $$A_{0v}(\underline{0},\underline{j};x)$$

$$= \int_0^x \sum_a^b \alpha_a p_{0v}(\underline{j},a;y)(1 - \exp(-(x-y)\sum\lambda))\prod_c^b \frac{\exp(-(\lambda_c y)(\lambda_c y)^{j_c}}{j_c!}\, dH_a(y)$$

ii) $u > 0$ or $i \neq 0$; again from corollary 3.5, equation (1) becomes

$$A_{uv}[\underline{i},h,s),(\underline{j},\cdot,\cdot);x] = \int_0^x \sum_k^b \int p_{uv}(\underline{e},r;y)\, f_{hk}(\underline{e},y,t|s)\, dt\, dH_r(y),$$

where $r = \min\{c\colon \delta_{uc} + i_c > 0,\ \ c = 1,2,\cdots,b\}$; using equation (2) and integrating out t, one obtains

$$(4)\qquad A_{uv}[(\underline{i},h,s),(\underline{j},\cdot,\cdot);x] = \int_0^x p_{uv}(\underline{e},r;y)\prod_c^b \frac{\exp(-(\lambda_c y)(\lambda_c y)^{e_c}}{e_c!}$$

since

$$\int_0^y \sum_h^b f_{hk}(\underline{e},y,t|s)dt = \prod_c^b \frac{\exp(-(\lambda_c y)(\lambda_c y)^{e_c}}{e_c!}$$

for $\underline{e} = [e_c]$ with $e_c = j_c + \delta_{rc} - \delta_{uc} - i_c > 0$. Since the right hand side of equation (4) does not depend on h and s, one can rewrite it as

$$(5)\qquad A_{uv}(\underline{i},\underline{j};x) = \int_0^x p_{uv}(\underline{e},r;y)\prod_c^b \frac{\exp(-(\lambda_c y)(\lambda_c y)^{e_c}}{e_c!}\, dH_r(y).$$

Equations (3) and (5) prove the proposition. □

Proposition 3.10. If $MR_b = M_b$, then the joint probability distribution of $Z'_n$ and $0_n$ is given by

$$\Pr\{Z'_n = k,\ 0_n \le x | Y_n = u,\ \underline{N}_{n-1} = \underline{i}\}$$

$$= \begin{cases} \alpha_k \int_0^x (1-\exp(-(x-y)\sum\lambda)dH_k(y)\ , & \text{if } u = 0 \text{ and } \underline{i} = \underline{0}, \\ \delta_{rk}H_k(x)\ , & \text{if } u > 0 \text{ or } \underline{i} \ne \underline{0} \end{cases}$$

where $r = \min\{c:\ \delta_{uc} + i_c > 0,\quad c = 1,2,\cdots,b\}$.

Proof. Use proposition 3.6 and lemma 3.7. □

Corollary 3.11. If $MR_b = M_b$, then $Z'_n$ depends only on $Y_{n-1}$ and $\underline{N}_{n-1}$ and its probability is given by

$$\Pr\{Z'_n = k \mid Y_{n-1} = u, \underline{N}_{n-1} = \underline{i}\} = \begin{cases} \alpha_k, & \text{if } u = 0 \text{ and } \underline{i} = \underline{0} \\ \delta_{rk}, & \text{if } u > 0 \text{ or } \underline{i} \neq \underline{0} \end{cases}$$

where $r = \min\{c: \delta_{uc} + i_c > 0,\ \ c = 1,2,\cdots,b\}$.

Proof. Use limit as x goes to $\infty$ in proposition 3.10. $\square$

Proposition 3.12. If $MR_b = M_b$, then $Y_n, Z'_n, \underline{N}_n, 0_n$ depend on $Y_{n-1}$ and $\underline{N}_{n-1}$. Moreover, its joint probability distribution is given by

$$\Pr\{Y_n = v, Z'_n = k, \underline{N}_n = \underline{j}, 0_n \leq x \mid Y_{n-1} = u, \underline{N}_{n-1} = \underline{i}\}$$

$$= \begin{cases} \alpha_k \displaystyle\int_0^x (1 - \exp(-(x-y)\textstyle\sum\lambda) P_{0v}(\underline{j},k;y) \displaystyle\prod_c^b \frac{\exp(-(\lambda_c y)(\lambda_c y)^{j_c}}{j_c!}\, dH_k(y), \\ \qquad\qquad\qquad\qquad\qquad \text{if } u = 0 \text{ and } \underline{i} = \underline{0}, \\ \delta_{rk} \displaystyle\int_0^x P_{uv}(\underline{e},r;y) \prod_c^b \frac{\exp(-(\lambda_c y)(\lambda_c y)^{e_c}}{e_c!}, \quad \text{if } u > 0 \text{ or } \underline{i} \neq \underline{0}; \end{cases}$$

where $\underline{e} = [e_c]$ with $e_c = j_c + \delta_{rc} - \delta_{uc} - i_c$ and $r = \min\{c: \delta_{uc} + i_c > 0,$ $c = 1,2,\cdots,b\}$.

Proof. Using propositions 3.9 and 3.10 it follows that $Y_n, Z'_n, N_n$ and $0_n$ depend only on $Y_{n-1}$ and $\underline{N}_{n-1}$. Depending on whether or not the system becomes idle just after $t_{n-1}$, there are two cases to consider:

i) $u = 0$ and $\underline{i} = \underline{0}$: here, one notices that to have $Z'_n = k$, the first arrival after $t_{n-1}$ must be a type-k unit. Therefore, following the development used in the proof of proposition 3.9, one obtains the first part of the desired result.

ii) $u > 0$ or $\underline{i} \neq \underline{0}$: here, the type of the next unit to serve is given by $Z'_n = r$ where

$$r = \min\{c: \delta_{uc} + i_c > 0,\ c = 1,2,\cdots,b\},$$

using assumption E. Similarly, one obtains the second part of the desired result. □

3.3. The Busy Cycle and Busy Period. *In this section*, the occupation of the server is analyzed. This occupation alternates between busy and idle periods. A busy period followed by an idle period constitutes a busy cycle. If $I_m$ denotes the length of the $m^{\underline{th}}$ idle period, the sequence $\{I_m\}$ was characterized by proposition 3.3.

Lemma 3.13. If $MR_b = M_b$, then the idle period process $\{I_m\}$ is Poisson with parameter $\sum\lambda$.

Proof. This is immediate. □

Now, turning our attention to the busy period process, let $B_m$ be the length of the $m^{\underline{th}}$ busy period length and $G(\cdot)$ its probability distribution under stationary conditions.

Theorem 3.14. If $MR_b = M_b$, then, under stationary conditions, the probability distribution of busy period lengths is given by

$$G^*(s) = \sum_r^b \alpha_r [L_{00}(s;r) + \sum_u^b \sum_v^b L_{0u}(s;r)(I-J(s))^{-1}_{uv} L_{v0}(s;v)],$$

where

$$L_{uv}(s;r) = \sum_{j \geq 0} \int_0^\infty \exp(-sy) p_{uv}(j,r;y) \prod_k^b \frac{\exp(-\lambda_k y)(\lambda_k y G^*(s))^{j_k}}{j_k!} dH_r(y);$$

and $J(s)$ is the matrix $[L_{uv}(s;u)]$ from which the first row and the first column have been removed.

Comment. The busy period is independent of the service discipline stated in assumption Ė. Moreover, the order of service is immaterial to the busy period as long as it does not increase the time spent in service. Hence, one can give priority to a fed back unit; that is, a unit is given successive periods of service until it departs from the system. If one calls the originator, the unit which initiates the busy period, the busy period of the system can be written as follows:

(1) $$B_m = S_n + S_{n+1} + \cdots + S_{n+F_m} + B_m^1 + \cdots + B_m^K ,$$

where

$S_{n+i}$ = the $(i+1)^{\underline{th}}$ service period of the originator,

$F_m$ = number of feedbacks of the originator,

$K = \sum_k^b N^k_{n+F_m}$ = number of units in the system when the originator departs,

$N^k_{n+i}$ = queue length of type-k units just after the $(i+1)^{\underline{th}}$ service completion of the originator.

$B^j_m$ = busy period initiated by the $j^{\underline{th}}$ arrival during the service periods of the originator.

The index n refers to a count on the number of services by the server. The service period $S_{n+i}$ depends only on the type $Z'_{n+i}$ for $i = 1,2,\cdots,F_m$ which in turn depends on $Y_{n+i-1}$. Hence, given the output type $Z_{n+i-1}$, the service periods $S_{n+i}$'s are independent. Furthermore, during the total period of service of the originator, $S_n + S_{n+1} + \cdots + S_{n+F_m}$, there have been K arrivals divided among b different types. Since the arrival process is $M_b$, the sequence of types of these K arrivals constitutes an independent process. Moreover, each of these K arrivals will be the originator of the busy period $B^j_m$ for some $j = 1,2,\cdots,K$. Therefore, all $B^j_m$'s are independent and have the same probability distribution that $B_m$ has, $G(\cdot)$. Moreover, each busy period $B^j_m$ is independent of the above service periods, $S_{n+1}$.

Proof (of theorem 3.14). If $Z_m$ denotes the type of the originator, then the probability distribution of the busy period $B_m$ is derived as follows:

$$G(x) = \Pr\{B_m \le x\}$$

$$= \sum_r^b \alpha_r \sum_k \Pr\{B_m \le x, F_m = k \mid Z_m = r\}$$

(1) $$= \sum_r^b \alpha_r \sum_k G_k(x;r).$$

Using equation (1) in the comments, the probability term for $k=0$ in equation (2) is

(2) $$G_0(x;r) = \Pr\{S_n + B_m^1 + \cdots + B_m^K \le x, Y_n = 0 \mid Y_{n-1} = 0, \underline{N}_{n-1} = 0, Z_m = r\}$$

$$= \sum_{\underline{j} \ge \underline{0}} \int_0^x G^{(h)}(x-y)p_{00}(\underline{j},r;y) \prod_k^b \frac{\exp(-(\lambda_k y)(\lambda_k y)^{j_k}}{j_k!} dH_r(y);$$

where $h = \sum_k^b j_k$. Considering the Laplace-Stieltjes transform of the probability distribution given in equation (3), one gets

(3) $$G_0^*(s;r) = L_{00}(s;r);$$

using an appropriate change of variables and the notation

(4) $$L_{uv}(s;r) = \sum_{\underline{j} \ge \underline{0}} \int_0^\infty \exp(-sy)\, p_{uv}(\underline{j},r;y) \prod_k^b \frac{\exp(-(\lambda_k y)(\lambda_k y G^*(s))^{j_k}}{j_k!} dH_r(y),$$

for every $u,v = 0,1,\cdots,b$.

For the case $k=1$, the term on the right hand side of equation (2) is

$$G_1(x;r) = \sum_v^b \Pr\{S_n + S_{n+1} + B_m^1 + \cdots + B_m^K - x, Y_n = v, Y_{n+1} = 0 \mid Y_{n-1} = 0,$$

$$\underline{N}_{n-1} = \underline{0}, Z_m = r\}$$

$$= \sum_{\underline{i} \ge \underline{0}} \sum_{\underline{j} \ge \underline{i}} \int_0^x \int_0^{x-y} \sum_v^b G^{(h)}(x-y-z)p_{v0}(\underline{j} - \underline{i},v;y)$$

$$\cdot \prod_k^b \frac{\exp(-(\lambda_k y)(\lambda_k y)^{j_k - i_k}}{(j_k - i_k)!} dH_v(z)p_{0v}(\underline{i},r;y)$$

$$\cdot \prod_k^b \frac{\exp(-(\lambda_k y)(\lambda_k y)^{i_k}}{i_k!} dH_r(y),$$

where $h = \sum_{k}^{b} j_k$. Now, *taking the Laplace-Stieltjes transform of the above* distribution, and by an appropriate change of variables, one finds

$$(5) \qquad G_1^*(s;r) = \sum_{v}^{b} L_{0v}(s;r)L_{v0}(s;v);$$

using notation given in equation (4).

Now, for $k = 2$, one could find

$$G_2^*(s;r) = \sum_{u}^{b}\sum_{v}^{b} L_{0u}(s:r)J_{uv}(s)L_{v0}(s;v),$$

where $J(s)$ is the matrix $[L_{uv}(s;u)]$ from which the first row and the first column has been removed. It also follows that for $k \geq 1$

$$(6) \qquad G_k^*(s;r) = \sum_{u}^{b}\sum_{v}^{b} L_{0u}(s:r)(J^{k-1}(s))_{uv}L_{v0}(s;v).$$

For a fixed $s$ with $Re\{s\} \geq 0$, the matrix $J(s)$ is such that $0 \leq L_{uv}(s;u) < 1$ and $\sum_{v}^{b} L_{uv}(s;u) < 1$, using the definition of $L_{uv}(s;u)$ and condition 1 of assumption C. Thus, all the eigenvalues of $J(s)$ have moduli less than 1, using a result from Lancaster [1969]. Therefore, $\sum_{k} J^k(s)$ converges and is equal to $(I-J(s))^{-1}$. Hence, summing equation (6) over $k$ from 1 to $\infty$, one obtains

$$\sum_{k}^{b} G_{k+1}^*(s;r) = \sum_{u}^{b}\sum_{v}^{b} L_{0u}(s;r)([I-J(s)]_{uv}^{-1} \quad L_{v0}(s;v),$$

together with equation (3), equation (4) gives the desired result. □

Notice that when the arrival process $MR_b$ is considered, the busy periods $B_m^j$ and $B_m^h$ for $j \neq h$ are no longer independent.

# 4. QUEUE LENGTHS, DEPARTURE PROCESSES, WAITING TIMES

4.0. Introduction. The purpose of this section is to expose some of the literature that explores various special cases of the general structure of sections 2 and 3. We will be principally concerned with those results for queue length processes, departure processes, and waiting time processes. Several previously obtained results occur by giving particular values to the parameters of the general model. We will not stop to point these out. They are noted in the referenced literature.

4.1. Queue Lengths. It is clear that sections 3.1, 3.2, 3.3 give the major tools one needs to work with the queue length processes of these systems. In particular, for example, proposition 3.9 gives the $\{Y_n, \underline{N}_n, 0_n\}$ structure in some detail. Clearly, $\{Y_n, \underline{N}_n\}$ is an embedded Markov chain for the $M_b/G_b/1$ queue with the feedback structure given in section 2.1. Thus, in principle, the embedded queue length process is a Markov process whose transition probabilities are now known. The study of the process is straightforward albeit quite cumbersome to execute.

Furthermore, again taking 3.2 as an example, $\{Y(t), \underline{N}(t)\}$ -- the continuous time queue length process for the $M_b/G_b/1$ queue with feedback -- is a semi-regenerative process in the sense of Çinlar [1975] and can be studied as such using the results of section 3.2.

d'Avignon and Disney [1976] study the embedded queue length process for the M/G/1 queue with the feedback structure given in section 2.1, but without $Z'_n$ dependence. As above, $\{Y_n, \underline{N}_n\}$ is a Markov chain. They give the one step transition probabilities and find necessary and sufficient conditions for the chain to be recurrent, non-null, aperiodic. The probability generating

function for the queue length process $\{Y_n, \underline{N}_n\}$ is then determined. The busy period and output process is studied in detail. The paper presents a few special cases of the results relevant to computer modelling. Of particular interest for applications is their study of the round robin system, the foreground-background problem, and the special case in which $\{Y_n\}$ is a Bernoulli process. This latter case seems to be the only one fully developed in previous feedback literature (e.g., see Takacs [1963] and the review of computer modelling uses of queues with feedback in Wyszewianski and Disney [1974]).

Disney and d'Avignon [1977] study the embedded queue length process for the $M_2/G_2/1$ queue with the feedback structure given in section 2.1 depending on $Z'_n$. They show $\{Y_n, \underline{N}_n\}$ is a Markov chain. They determine conditions necessary and sufficient for the $\{Y_n, \underline{N}_n\}$ process to have a steady state and then determine the probability generating functions for the joint $\{Y_n, N_n^1, N_n^2\}$ process. $\{Y_n, Z'_n\}$ is studied as are several marginal processes obtained from these. The busy period is studied in detail.

Fujisawa and Osawa [1975], in the course of studying departures from queues with feedback, consider the M/M/1 queue with $\{Y_n\}$, a two state Markov process. They show that $\{Y_n, \underline{N}_n\}$ is a Markov chain, find the one step transition probability matrix, conditions necessary and sufficient for the Markov chain to have a limiting distribution, and give the generating function for that limiting distribution.

In each of the above cases, the derived generating functions seem to be quite complex. We have not reproduced the results in this paper for that reason. The interested reader will find details in the referenced paper.

In studying these processes in detail, it quickly becomes apparent that

in many cases one can consider a queueing system without feedback whose asymptotic queue length distribution is identical to that for a queue with feedback. Indeed, these queues with feedback have often been dismissed from examination precisely for this reason. d'Avignon and Disney [1978] explore this question in more detail. By studying the queue length process at departure epochs (instead of output epochs as is done herein), one can obtain a queue length process for a queue without feedback that is asymptotically identically distributed. The cases they study (i.e. $\{Y_n\}$ has the structure of section 2.1 without the dependence on $Z_n'$ and with Poisson arrivals) give sufficient conditions to ensure this result. We know of no more general result.

4.2. Departures. In these queues with feedback there are at least two ways to study the departure process. One can filter the output process or one can study the processes embedded at departure points. The former method is used in d'Avignon and Disney [1976] for the system discussed in section 4.1. The second method is used in the two Fujisawa et al. papers ([1974;1975]) noted in section 4.1. This second method is also the approach used in d'Avignon and Disney [1978] to study that problem (see section 4.1).

Disney and McNickle present a study of the M/G/1 queue with $\{Y_n\}$ as a Bernoulli process (this is the problem studied by Takacs [1963]) with particular reference to the output process, departure process, input process, and feedback process. Their departure results are relevant here. In their system (as in d'Avignon and Disney [1978]) it is shown that the departure process is a Markov renewal process. The relevant transition functions are

given therein. The departure process is a Poisson process only for $G = M$. Their results show that the output process from their queues are never renewal processes except in the special case in which there is no feedback (i.e. $\Pr[Y_n = 0] = 1$)and, in that case, one obtains a renewal process (which is a Poisson process) only if $G = M$ (which is in agreement with the Burke [1956;1976] or Disney, et al. [1973] results).

Melamed [1977] presents a study of Jackson queueing networks (Jackson [1957]) and is able to show that if there is one node in the network with the property that output from that node has a non-zero probability of eventually returning to that node (i.e. $\theta_{ii}^{(n)} > 0$ for some $n < 0$ using Jackson's symbolism), then flow processes along that loop are never Poisson processes. This result adds new interest to the often used Jackson network results.

4.3. Waiting Times. Waiting times in these queues with feedback seem to be a major problem. One can consider many such waiting time problems depending on where the fedback unit re-enters the queue. If the feedback returns to the head of the queue, then from the d'Avignon and Disney results [1978] noted in section 4.1, one can find a waiting time problem equivalent to a waiting time problem for a queue without feedback. However, even in the cases considered there, if the feedback reappears anywhere else in the line (e.g. the feedback goes to the end of the line as is a common assumption), the waiting time problem is formidable. The problem is caused by the fact that the waiting time in the system is the sum of the waiting times that that customer spends each time he goes through the server. Unfortunately, the waiting times on each pass through the server for the same customer are not independent of each other. This is easily seen by noting that the

waiting time for a customer going through the system for a second time depends on the number of customers to enter the system while he was waiting to go through on his first pass.

Takacs [1963] is the only paper that we are aware of that studies the waiting time problem in detail. His results are for M/G/1 queues with $\{Y_n\}$ a Bernoulli process. Wyszewianski and Disney [1974] surveys the computer literature and presents results therein concerning moments of the waiting time distribution for round robin and foreground-background models.

A point ofter made concerning the queue length process for the M/G/1 queue with $\{Y_n\}$ a Bernonlli process is that it is equivalent to a queue without feedback in the sense that there always exists an M/G/1 queue without feedback whose limiting probability distribution (for queue length) is equal to that for the M/G.1 queue with Bernonlli feedback. The question this arises as to whether there is an M/G/1 queue without feedback whose limiting sojourn time distribution is equal to that of the M/G/1 queue with Bernonlli feedback. The next lemma is a partial answer to the question.

Lemma 4.1. If the feedback unit goes to the tail of the line and if the M/G/1 queue without feedback and the M/G/1 queue with Bernoulli feedback are to have the the same first three moments for their respective total sojourn time distributions, then there is no M/G/1 queue without feedback whose limiting sojourn time distribution is equal to that of the M/G/1 queue with Bernoulli feedback.

Proof. Let $T^f$ be the sojourn time in the queue with feedback to the tail of the line and $T^w$ the corresponding sojourn time for the queue without feedback. Let $\alpha_n$ be the $n^{th}$ moment about 0 of the service time (for one pass through the server) of the feedback queue. Let $\tau_n$ be the $n^{th}$ moment about 0 for the total service time in the queue without feedback. If the

two queues are to have the same first three moments than it is a simple exercise to show

$$\tau_1 = \alpha_1/q$$

$$\tau_2 = \alpha_2/q + 2(1-q)\alpha_1^2/q^2$$

$$\tau_3 = \alpha_3/q + 6(1-q)\alpha_1\alpha_2/q^2 + 6(1-q)^2\alpha_1^3/q^3$$

Then using the expected sojourn time and the second mement of the sojourn time for the queue with feedback given by Takacs [1963] and the well known results of corresponding moments for the M/G/1 queue without feedback one obtains

$$E[T^w] = E[T^f]$$

$$E[(T^w)^2] \neq E[(T^f)^2]$$

unless $q = 1$(i.e. there is no feedback) □

The following results structure the problem as a Markov renewal process for the M/M/1 queue with $\{Y_n\}$ a Bernoulli process, but even here computations using this structure seem too tedious. We have not pursued the topic.

Pick a customer once and for all. Call him C. Suppose he feeds back K times in all. The distribution of K is geometric in this case. Let

$N_n$ = the number of customers left behind the $n^{\underline{th}}$ time C outputs $(n = 1,2,\cdots)$

$T_n$ = the time at which C outputs for the $n^{\underline{th}}$ time $(n = 1,2,\cdots)$

$X_n = T_n - T_{n-1}$ (the sojourn time of C on his $n^{\underline{th}}$ pass), $(n = 2,3,\cdots)$

$T_1 = X_1$ = soujourn time of C on his $1^{\underline{st}}$ pass through service

$B_n(y)$ = number of exogeneous arrivals in the $n^{\underline{th}}$ sojourn time of C, $(n = 1,2,\cdots)$. $B_n(y)$ is a Poisson process ($\lambda$) for every n. B (y) is a Poisson process ($\lambda$) for every n.

$C_n$ = number of customers ahead of C at the start of his $n^{\underline{th}}$ pass through the server who feedback - $C_n$ is a bimomial random variable given $N_{n-1}$.

Theorem 4.2 $\{N_n, X_n\}$ is a delayed Markov renewal process.

Proof. This is quite apparent. □

Let $H_c(x)$ be the remaining service time of the customer in service at the arrival of C. Let $H_i(x)$ be the i-fold convolution of H with itself. Then we have

Corollary 4.3. The transition functions for the $\{B_b m X_n\}$ process are given by:

$$D_{ij}(x) = \Pr[N_1=j, X_1 \le x \mid N_o=i]$$

$$= \int_o^x H(dy)A_{oj}(y), \qquad \text{if } i = o,$$

$$= \int_o^x (H_c * H_i)(dy)A_{ij}(y), \qquad i = 1,2,\cdots.$$

and

$$Q_{ij}(x) = \Pr[N_n=j, X_n \le x \mid N_{n-1}=i]$$

$$= \int_o^x H(dy)A_{oj}(y), \qquad i = o,$$

$$= \int_o^x H_{i+1}(dy)A_{ij}(y), \qquad i = 1,2,\cdots.$$

Here

$$A_{ij}(y) = \begin{cases} \sum_{m=o}^{j} \binom{i}{m} p^m q^{i-m} (\lambda y)^{j-m} e^{-\lambda y}/m!, & j \le i \\ \sum_{n=o}^{i} \binom{i}{m} p^m q^{i-m} (\lambda y)^{j-m} e^{-\lambda y}/m!, & j > i. \end{cases}$$

Proof. Again this is apparent. □

Then since $\{N_n, X_n\}$ is a Markov renewal process we have

Corollary 4.4.

$$Q_{ij}^{(k)}(y) = Pr\left[N_k = j, T_k^f \le y \mid N_o = i\right]$$

$$= \int_o^y \sum_{k=1}^{\infty} D_{im}(dx) Q_{mj}^{(k-1)}(y-x)$$

Proof. These are the usual k step transition functions for a delayed Markov renewal process. □

Then if there exists a vector $\Pi$ satisfying

$$\Pi = \Pi\, Q(\infty)$$

we have .

Theorem 4.5. The sojourn time for C is given by

$$Pr\left[T^f \le y\right] = \Pi \left[ \sum_{k=1}^{\infty} Q_{ij}^{(k)}(y) p^{i-1} q \right] U.$$

where U is the column vectors whose elements are all 1.

<u>Proof</u>. Conditioned on C feeding back k times corollary 4.4 gives the joint distribution of $N_k$ and $T_k^f$. conditioned on $N_o$. The result then follows by removing the condition for the number of feedbacks and for $N_o$ and finding the resulting marginal distribution for $T^f$. □

Thus the marginal sojourn time of C are given by theorem 4.5. Notice that, in general, successive sojourn times are not independent and therefore the sojourn time process is not specified by theorem 4.5 alone. Notice also that the Bernoulli process $\{Y_n\}$ plays a minor role. The results can be generalized at least to the case

$$\Pr\left[Y_n=u \mid Y_{n-1}=v,\ N_n=j, N_{n-1}=i,\ S_n=y\right].$$

Thus in principle, the sojourn time problem for M/G/1 queues with feedback is solved for a rather large class of $\{Y_n\}$ processes. To go beyond theorem 4.5 , however, to find $\Pr\left[T^f\right]$ explicitly seems to be a formidable computation problem even for M/M/1 queues with Bernoulli feedback.

There is an interesting question in these results, however. We know from lemma 4.1 that there is no M/G/1 queue without feedback whose sojourn time distribution is equal to the M/G/1 queue with Bernoulli feedback if the two service times are to have the same first three moments. If we relax this moment requirement, the problem can be exposed as follows.

Let $^*Q(x)$ be any Markov renewal kernel for which there exists a probability vector $^*\Pi$ such that

$$^*\Pi\, ^*Q(\infty) = {}^*\Pi.$$

Then is there a $(^*\Pi, {}^*Q)$ pair for which

$$\Pi\left[\sum_{k=1}^{\infty} Q^{(k)}(x)p^{k-1}q\,U\right] = {}^*\Pi\, ^*Q(x)?$$

If there is such a pair, is there an M/G/1 queue without feedback for which *Q(x) is the Markov renewal kernel for the sojourn time of a customer? That is, is there any M/G/1 queue without feedback whose sojourn time distribution for a given customer is equal to that sojourn time given for the M/G/1 queue with Bernoulli feedback in theorem 4.5?

Acknowledgements

We would like to thank Robert Foley and Burton Simon of the Department of Industrial Engineering and Operations Research of Virginia Polytechnic Institute and State University for their many long discussions on various aspects of this paper.

REFERENCES

[1] Burke, P. J. (1956) "The Output of a Queueing System", Oper. Res., 4, 699-704.

[2] Burke, P. J. (1976) "Proof of a Conjecture on the Interarrival Time Distribution in an M/M/1 Queue with Feedback", IEEE Transactions on Communications, 24, 575-576.

[3] Çinlar, E. (1975) An Introduction to Stochastic Processes, Prentice-Hall, New York.

[4] d'Avignon, G. R. and Disney, R. L. (1976) "Single Server Queues with State-Dependent Feedback", INFORS (Canada), 14, 71-85.

[5] d'Avignon, G. R. and Disney, R. L. (1977) "Queues with Instantaneous Feedback", Mgmt. Sci., 24, 168-180.

[6] d'Avignon, G. R. and Disney, R. L. (1978) "Equivalence for Feedback Queues with Poisson Arrivals" (submitted for publication).

[7] Disney, R. L. (1975) "Random Flow in Queueing Networks: A Review and Critique", A.I.I.E. Transactions, 7, 268-288.

[8] Disney, R. L. and McNickle, D. (1977) "The M/G/1 Queue with Instantaneous Feedback", Technical Report 77-4, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, Michigan (submitted for publication).

[9] Fujisawa, T. and Chen, Y. (1974) "The Output Process of the Queueing Systems with Feedback", Rep. Univ. Electro-Comm., 25-1, (Sci. and Tech. Sect.), 75-78.

[10] Fujisawa, T. and Osawa, H. (1975) "The Departure Process of the M/M/1 Queue with Markovian Feedback", Rep. Univ. Electro-Comm., 25-2, (Sci. and Tech. Sect.), 193-196.

[11] Jackson, J. R. (1957) "Networks of Waiting Lines", Oper. Res., 15, 254-265.

[12] Kleinrock, L. (1975) Queueing Systems Volume I: Theory, Wiley, New York.

[13] Kleinrock, L. (1976) Queueing Systems Volume II: Computer Applications, Wiley, New York.

[14] Kopocinski, B. (1963) "On a Model of a Queue with Feedback", Zast. Math. Appl. Math., 7, 51-57.

[15] Kopocinska, I. (1967) "On a M/G/1 Queueing Model with Feedback", Zast. Math. Appl. Math., 9, 161-171.

[16] Kopocinska, I. and Kopocinski, B. (1972) "Queueing Systems with Feedback", Zast. Math. Appl. Math., 12, 373-384.

[17] Lanchester, P. (1969) Theory of Matrices, Academic Press, New York.

[18] McFadden, J. A. (1962) "On the Lengths of Intervals in a Stationary Point Process", J. Royal Stat. Soc., B, 24, 364-382.

[19] Melamed, B. (1977) "Characterization of Poisson Traffic Streams in Jackson Queueing Networks" (submitted for publication).

[20] Takacs, L. (1963) "A Single Server Queue with Feedback", Bell Systems Tech. J., 42, 505-519.

[21] Vlach, T. L. and Disney, R. L. (1969) "The Departure Process from the GI/G/1 Queue", J. Appl. Prob., 6, 704-707.

[22] Wyszewianski, R. J. and Disney, R. L. (1974) "Feedback Queues in Modelling of Computer Systems: A Survey", Technical Report 74-1, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, Michigan.